A NOTE ON COMPLETE SUBDIVISIONS IN DIGRAPHS OF LARGE OUTDEGREE

DANIELA KÜHN, DERYK OSTHUS, AND ANDREW YOUNG

Abstract. Mader conjectured that for all ℓ there is an integer $\delta^+(\ell)$ such that every digraph of minimum outdegree at least $\delta^+(\ell)$ contains a subdivision of a transitive tournament of order ℓ. In this note we observe that if the minimum outdegree of a digraph is sufficiently large compared to its order then one can even guarantee a subdivision of a large complete digraph. More precisely, let $\vec{G}$ be a digraph of order n whose minimum outdegree is at least d. Then $\vec{G}$ contains a subdivision of a complete digraph of order $\lfloor d^2/(8n^{3/2})\rfloor$.

1. Introduction

A fundamental result of Mader [4] states that for every integer ℓ there is a smallest $d = d(\ell)$ so that every graph of average degree at least d contains a subdivision of a complete graph on ℓ vertices. Bollobás and Thomason [1] as well as Komlós and Szemerédi [3] showed that $d(\ell)$ is quadratic in ℓ. In [6], Mader made the following conjecture, which would provide a digraph analogue of these results (a transitive tournament is a complete graph whose edges are oriented transitively).

Conjecture 1 (**Mader** [6])**.** *For every integer $\ell > 0$ there is a smallest integer $\delta^+(\ell)$ such that every digraph $\vec{G}$ with minimum outdegree at least $\delta^+(\ell)$ contains a subdivision of the transitive tournament on ℓ vertices.*

It is easy to see that $\delta^+(\ell) = \ell - 1$ for $\ell \leq 3$. Mader [7] showed that $\delta^+(4) = 3$. Even the existence of $\delta^+(5)$ is not known. One might be tempted to conjecture that large minimum outdegree would even force the existence of a subdivision of a large complete digraph (a complete digraph has a directed edge from v to w for any ordered pair v, w of vertices). However, for all n Thomassen [9] constructed a digraph on n vertices whose minimum outdegree is at least $\frac{1}{2}\log_2 n$ but which does not contain an even directed cycle (and thus no complete digraph on 3 vertices). The additional assumption of large minimum indegree in Conjecture 1 does not help either. Mader [6] modified the construction in [9] to obtain digraphs having arbitrarily large minimum indegree and outdegree without a subdivision of a complete digraph on 3 vertices.

The fact that one certainly cannot replace the minimum outdegree in Conjecture 1 by the average degree is easy to see: consider the complete bipartite graph with equal size vertex classes and orient all edges from the first to the second class. The resulting digraph $\vec{B}$ has average degree $|\vec{B}|/2$ but not even a directed cycle or a transitive tournament on 3 vertices. (On the other hand, Jagger [2] showed that if the average degree of a digraph $\vec{G}$ is a little larger than $|\vec{G}|/2$, then $\vec{G}$ does contain a subdivision of a large complete digraph.)

So in some sense, the above examples and constructions show that Conjecture 1 is the only possible analogue of the result in [4] mentioned above. Our main result is that if the minimum outdegree of a digraph is sufficiently large compared to its order, then Conjecture 1 is true. In fact, we show that in this case, one can even guarantee a subdivision of a complete digraph.

Theorem 2. *Let $\vec{G}$ be a digraph of order n whose minimum outdegree is at least d. Then $\vec{G}$ contains a subdivision of the complete digraph of order $\lfloor d^2/(8n^{3/2})\rfloor$.*

Note that the bound is nontrivial as soon as d is a little larger than $n^{3/4}$. Also, recall that the result of Thomassen [9] mentioned above implies that we cannot have a subdivision of a complete digraph of order at least 3 if $d \leq \frac{1}{2}\log_2 n$. Furthermore, note that if $d = cn$, then Theorem 2 guarantees a subdivision of a complete digraph of order $\lfloor c'\sqrt{n}\rfloor$, where $c' = c^2/8$. It is easy to see that this is best possible up to the value of c' (consider the complete bipartite digraph with vertex classes of equal size).

The main ingredient in the proof of Theorem 2 is Lemma 4. It states that if $\vec{G}$ has n vertices and its minimum outdegree is $\gg \sqrt{n}$, then $\vec{G}$ has a subdigraph $\vec{H}$ which is highly connected in the following sense: if x is any vertex of $\vec{H}$ and y is a vertex of large indegree, then there are many internally disjoint dipaths from x to y in $\vec{H}$. Lemma 4 also guarantees the existence of many such vertices y. For undirected graphs, there is a much stronger result of Mader [5] which implies that every graph of minimum degree at least $4k$ has a k-connected subgraph. Since a digraph version of this result is not known, Lemma 4 may be of independent interest. There are also several related results of Mader [6, 8] which investigate the existence of pairs of vertices with large local connectivity in digraphs of large minimum outdegree. The proof of Lemma 4 is quite elementary: if the current subdigraph $\vec{H}$ does not satisfy the requirements, then we can use Menger's theorem to find a significantly smaller subdigraph whose minimum outdegree is almost as large as that of $\vec{H}$. Since this means that the density of the successive subdigraphs increases, this process must eventually terminate.

2. Proof of Theorem 2

Before we start with the proof of Theorem 2 let us introduce some notation. The digraphs $\vec{G}$ considered in this note do not contain loops and between any ordered vertex pair $x, y \in \vec{G}$ there is at most one edge from x to y. (There might also be another edge from y to x.) We denote by $\delta^+(\vec{G})$ the minimum outdegree of a digraph $\vec{G}$ and by $|\vec{G}|$ its order. We write $d^+_{\vec{G}}(x)$ for the outdegree of a vertex $x \in \vec{G}$ and $d^-_{\vec{G}}(x)$ for its indegree. A digraph $\vec{H}$ is a *subdivision* of $\vec{G}$ if $\vec{H}$ can be obtained from $\vec{G}$ by replacing each edge $\vec{xy} \in \vec{G}$ with a dipath from x to y such that all these dipaths are internally disjoint for distinct edges. The vertices of $\vec{H}$ corresponding to the vertices of $\vec{G}$ are called *branch vertices*.

Given two vertices x and y of a digraph $\vec{G}$, we define $\kappa_{\vec{G}}(x, y)$ to be the largest integer $1 \leq k \leq |\vec{G}| - 2$ such that $\vec{G} - S$ contains a dipath from x to y

for every vertex set $S \subseteq V(\vec{G}) \setminus \{x, y\}$ of size $< k$. We define $\kappa_{\vec{G}}(x, y) := 0$ if $\vec{G}$ does not contain a dipath from x to y. We will use the following version of Menger's theorem for digraphs.

Theorem 3 (**Menger's theorem for digraphs**). *Let x and y be vertices of a digraph $\vec{G}$ such that $\kappa_{\vec{G}}(x, y) \geq k$. Then $\vec{G}$ contains k internally disjoint dipaths from x to y.*

As mentioned above, the main step in the proof of Theorem 2 is to find a subdigraph $\vec{H}$ of $\vec{G}$ such that the minimum outdegree of $\vec{H}$ is still large and such that every vertex of $\vec{H}$ sends many internally disjoint dipaths to each vertex of $\vec{H}$ which has large indegree.

Lemma 4. *Every digraph $\vec{G}$ of order n with $\delta^+(\vec{G}) \geq d$ contains a subdigraph $\vec{H}$ such that*

(i) $\delta^+(\vec{H}) > d/2$,
(ii) $\kappa_{\vec{H}}(x, y) \geq d^2/(4n)$ *for all pairs* $x, y \in V(\vec{H})$ *with* $d^-_{\vec{H}}(y) \geq d/2$,
(iii) *at least* $d^2/(4n)$ *vertices of* $\vec{H}$ *have indegree at least* $d/2$ *in* $\vec{H}$.

Proof. Put

$$\alpha := \frac{d}{n} \quad \text{and} \quad \alpha' := \frac{d^2}{4n^2} = \frac{\alpha^2}{4}.$$

By Theorem 3 we may assume that $\kappa_{\vec{G}}(x, y) < \alpha' n$ for some vertices x, y of $\vec{G}$ with $d^-_{\vec{G}}(y) \geq d/2$. Otherwise we could take $\vec{H} := \vec{G}$. (It is easy to check that $\vec{H}$ then also satisfies condition (iii) of the lemma.) Let $S \subseteq V(\vec{G}) \setminus \{x, y\}$ be a set of size $< \alpha' n$ such that $\vec{G} - S$ does not contain a dipath from x to y. Let Y be the set of all those vertices z for which $\vec{G} - S$ contains a dipath from z to y. Then $Y \cup S$ contains y as well as all the at least $d/2 = \alpha n/2$ inneighbours of y. Let C denote the component of the undirected graph corresponding to $\vec{G} - (Y \cup S)$ which contains x. Let $\vec{G}_1$ be the subdigraph of $\vec{G}$ induced by all vertices in C. Then $|\vec{G}_1| \leq n - |Y \cup S| < (1 - \alpha/2)n$. Moreover, note that there exists no edge directed from a vertex of $\vec{G}_1$ to a vertex outside $V(\vec{G}_1) \cup S$. Thus

$$\delta^+(\vec{G}_1) \geq \delta^+(\vec{G}) - |S| > (\alpha - \alpha')n. \tag{1}$$

If $\vec{G}_1$ does not satisfy condition (ii) of the lemma we again apply Theorem 3 to obtain a subdigraph $\vec{G}_2 \subseteq \vec{G}_1$. We continue in this fashion until we obtain a subdigraph $\vec{G}_r$ which satisfies condition (ii). We will show that $\vec{G}_r$ also satisfies (i) and (iii). Put $\vec{G}_0 := \vec{G}$,

$$\delta_i := \frac{\delta^+(\vec{G}_i)}{|\vec{G}_i|} \quad \text{and} \quad \gamma_{i-1} := \frac{|\vec{G}_{i-1}|}{|\vec{G}_i|}$$

for all $i \leq r$. Similarly as in (1) it follows that

$$\delta^+(\vec{G}_i) = \delta_i |\vec{G}_i| \geq \delta_{i-1} |\vec{G}_{i-1}| - \alpha' n \geq (\alpha - i\alpha')n. \tag{2}$$

Thus $\delta_i \geq \delta_{i-1}\gamma_{i-1} - \alpha' n/|\vec{G}_i| = \delta_{i-1}\gamma_{i-1} - \alpha' \prod_{j=0}^{i-1}\gamma_j$. Using this inequality and induction on i one can show that

$$\delta_i \geq (\alpha - i\alpha')\prod_{j=0}^{i-1}\gamma_j = (\alpha - i\alpha')\frac{n}{|\vec{G}_i|}. \tag{3}$$

Since we delete at least $d/2 = \alpha n/2$ vertices when going from $\vec{G}_{i-1}$ to $\vec{G}_i$ (namely the inneighbours of the vertex playing the role of y), we have that $|\vec{G}_r| \leq n - r\alpha n/2$. In particular this shows that $r < 2/\alpha$. However, since (3) implies that $1 > \delta_r \geq (\alpha - r\alpha')/(1 - r\alpha/2)$ we even have $r < (1-\alpha)/(\alpha/2 - \alpha')$. Thus

$$\delta^+(\vec{G}_i) \overset{(2)}{\geq} (\alpha - r\alpha')n \geq \left(\alpha - \frac{1-\alpha}{2/\alpha - 1}\right)n = \frac{\alpha n}{2-\alpha} > \frac{d}{2}. \tag{4}$$

Altogether this shows that $\vec{G}_r =: \vec{H}$ satisfies conditions (i) and (ii) of the lemma. To check that $\vec{H}$ also satisfies condition (iii) let ℓ denote the number of vertices of indegree $\geq d/2$ in $\vec{H}$. Then

$$\frac{\alpha n|\vec{H}|}{2-\alpha} \overset{(4)}{\leq} \delta^+(\vec{H})|\vec{H}| \leq |\vec{H}|\frac{d}{2} + \ell|\vec{H}|,$$

which implies that $\ell \geq \alpha d/(4-2\alpha) \geq d^2/(4n)$, as required. □

Proof of Theorem 2. Let $\ell := \lfloor d^2/(8n^{3/2})\rfloor$. We first apply Lemma 4 to obtain a subdigraph $\vec{H} \subseteq \vec{G}$ as described there. We pick a set $X \subseteq V(\vec{H})$ of ℓ vertices having indegree $\geq d/2$ in $\vec{H}$. (Such a set X exists by condition (iii) of Lemma 4.) X will be the set of our branch vertices. For every pair $x, y \in X$ there exist at least $d^2/(4n)$ internally disjoint dipaths from x to y. Thus the average number of inner vertices on such a path is at most $4n^2/d^2$. Hence $\vec{H}$ contains at least $d^2/(8n)$ internally disjoint dipaths from x to y such that each of these has at most $8n^2/d^2$ inner vertices. Let us call such a dipath *short*. This shows that we can connect all pairs of branch vertices greedily (in both directions) by choosing each time a short dipath which is internally disjoint from all the short dipaths chosen before. In each step we destroy at most $8n^2/d^2$ further dipaths. But $(|X|^2-1)8n^2/d^2 < 8\ell^2n^2/d^2 \leq d^2/(8n)$, so we can connect all pairs of branch vertices by short dipaths. □

References

[1] B. Bollobás and A. Thomason, Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs, *European Journal of Combinatorics* **19** (1998), 883–887.

[2] C. Jagger, Extremal Digraph Results for Topological Complete Subgraphs, *European Journal of Combinatorics* **19** (1998), 687–694.

[3] J. Komlós and E. Szemerédi, Topological Cliques in Graphs II, *Combinatorics, Probability and Computing* **5** (1996), 70–90.

[4] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, *Math. Annalen* **174** (1967), 265–268.

[5] W. Mader, Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend großer Kantendichte, *Abh. Math. Sem. Univ. Hamburg* **37** (1972), 86–97.

[6] W. Mader, Degree and Local Connectivity in Digraphs, *Combinatorica* **5** (1985), 161–165.
[7] W. Mader, On Topological Tournaments of order 4 in Digraphs of Outdegree 3, *Journal of Graph Theory* **21** (1996), 371–376.
[8] W. Mader, Existence of vertices of local connectivity k in digraphs of large outdegree, *Combinatorica* **15** (1995), 533–539.
[9] C. Thomassen, Even Cycles in Directed Graphs, *European Journal of Combinatorics* **6** (1985), 85–89.

Daniela Kühn, Deryk Osthus & Andrew Young
School of Mathematics
University of Birmingham
Edgbaston
Birmingham
B15 2TT
UK
E-mail addresses: {kuehn,osthus,younga}@maths.bham.ac.uk